Int. J. Solids Structures, 1974, 10, pp. 231–242. Pergamon Press. Printed in Gt. Britain.

DISCRETE LATTICE SIMULATION OF TRANSIENT MOTIONS IN ELASTIC AND VISCOELASTIC COMPOSITES

ADNAN H. NAYFEH

Systems, Science and Software, P.O. Box 1620, La Jolla, California 92037

(*Received* 1 *May* 1973; *revised* 8 *August* 1973)

Abstract—Discrete viscous lattices are studied in order to mathematically simulate the propagation of general disturbances in semi-infinite, elastic and viscoelastic periodic composites. Integral transform techniques are employed to derive exact and approximate solutions for the cases in which viscosity is introduced to the elastic lattice by adding dash pots in both series and parallel with the springs connecting adjacent particles. Solutions for the continuum models (for vanishing lattice spacings) are also derived either directly or as special cases. It is demonstrated that the effect of the periodic discretization in these systems induces oscillation about the continuum solutions. Comparison of the exact and approximate results shows excellent correlation, particularly near the wave front.

1. INTRODUCTION

A review of the existing literature on structural composites reveals no exact analytical solutions for the propagation of general disturbances (transient motions) on periodic composites. Approximate transient solutions for a wide variety of periodic composites have been derived, however, by Fourier synthesizing pulses from the low frequency portions of the composite's dispersion relations. The resulting solutions are judged to be valid only in the so-called "far field," and have been termed the "Head of the Pulse" approximation[1]. Assessment of the ranges of validity of these solutions has been the subject of some investigations.

With this in mind, Drumheller and Sutherland[2] recently pointed out that the geometric dispersion observed in a wide variety of composites is believed to result mainly from the spatial periodicity rather than from the precise shape of the reinforcing elements. On the basis of this observation, and in the interest of studying the transient response of periodic composites to externally applied pulses, these authors suggested modeling such composites as distributions of periodically laminated plates. They refer to the laminated plates as "continuous" lattices as opposed to "discrete" lattices of point masses and springs.

Probably the simplest periodic system is the discrete lattice made up of equal masses connected by identical elastic springs as shown in Fig. 1(a). While exact dispersion relations for the propagation of harmonic waves on discrete lattices have been known for a long time[3], exact transient motions of such lattices are relatively new, especially those of the semi-infinite chains (see for example [4–6]). The exact transient solutions obtained in [4–6] are observed to exhibit the same general behavior as the far field approximate solutions of the elastic composites (see for example [7–9]).

Far field solutions similar to those derived in [7, 8] can also be obtained for the discrete lattice simply by utilizing a Taylor series expansion in terms of small lattice spacing. The resulting approximate solutions are found to correlate well with exact results especially near the wave front. With the lack of exact transient solutions for the composites, this correlation, together with the many basic similarities between the lattice and the periodic composites (see [2]), provide additional impetus to studying more complicated lattices in order to qualitatively investigate the influence of other physical effects on composites. One such effect which will be discussed in detail in the following, is viscosity.

As distinguished from the geometric dispersion mentioned above, pulse profile data on a number of composites have also shown a strong evidence of viscoelastic behavior[10]. Sve[11] has investigated the influence of viscosity on the far field response of composites by allowing the wave number to be complex. In this paper the transient motion of viscoelastic composites is mathematically simulated by studying the corresponding response of the semi-infinite monoatomic lattice which includes a dissipative mechanism. Exact and asymptotic solutions are obtained for the cases in which viscosity is introduced by adding dash pots in series (Maxwell model) and parallel (Voigt model) with the springs connecting adjacent particles as shown in Figs. 1(b) and 1(c), respectively.

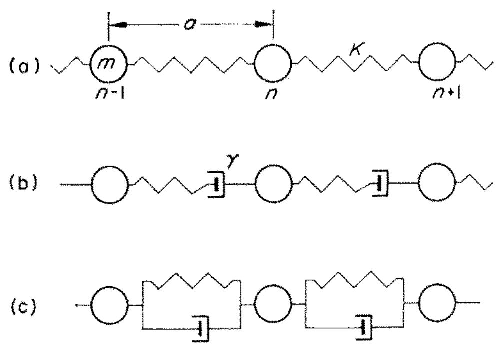

Fig. 1. The lattice models.

At this stage it is appropriate to indicate that the present analysis *does not* suggest modeling periodic composites as discrete lattices. It is merely intended to assess (through the correlation of exact and approximate solutions of lattices) the validity of the composite's approximate solutions. One may suggest, however, that a better simulation of a composite, such as the laminated plates, may be constructed by considering a diatomic lattice. An attempt to consider transient motions on such a lattice reveals that, as in the case of composites, analytic solutions are quite difficult to obtain. One can obtain asymptotic solutions, however, which can be shown to behave not unlike the asymptotic solutions of the monatomic chain. Thus, without any loss in mathematical generality, it often suffices to treat only the case of the monatomic lattice.

2. THE PURELY ELASTIC LATTICE

Before proceeding to study the influence of viscosity, we give a quick review of the purely elastic problem solved in [6]. The mathematical problem treated in [6] is

$$\ddot{u}_n = \omega^2(u_{n+1} - 2u_n + u_{n-1}), \qquad n = 1, 2, 3, \ldots, \tag{1a}$$

$$u_0(t) = \phi(t), \tag{1b}$$

$$u_n(0) = \dot{u}_n(0) = 0, \tag{1c}$$

where $\omega^2 = k/m$, k is the spring constant, m is the particle mass, superposed dots stand for time derivatives, and $\phi(t)$ is a prescribed function of time.

The solution to this problem is found by taking Laplace transform with respect to time, followed by setting

$$\bar{u}_n(p) = \bar{\phi}(p)X^n(p), \tag{2}$$

where $\bar{u}_n(p)$ is the Laplace transform of $u_n(t)$, to obtain the characteristic equation

$$X^2 - 2[(p^2/2\omega^2) + 1]X + 1 = 0. \tag{3}$$

Equation (3) has two roots; the relevant one is

$$X = [(p/2\omega) - |(p/2\omega)^2 + 1]^{1/2}|^2 \tag{4}$$

which, if substituted into (2) and Laplace transform inversion tables are used, yields

$$u_n(t) = \mathscr{L}^{-1}\left[\bar{\phi}(p)\left\{\frac{p}{2\omega} - \left[\frac{p^2}{4\omega^2} + 1\right]^{1/2}\right\}^{2n}\right] \tag{5a}$$

$$= 2n\int_0^t \phi(t-\tau)J_{2n}(2\omega\tau)\,\frac{\mathrm{d}\tau}{\tau}, \tag{5b}$$

where J_{2n} represents the Bessel function of the first kind.

The integral (5b) was evaluated numerically for $n = 10$, $\omega = 0{\cdot}5$ and the response for the constant velocity boundary condition ($\phi(t) = Vt$) vs time is labeled by $\alpha = 0$ in Fig. 2

3 DASH POTS IN SERIES

Considering only nearest-neighbor-interactions, we write the equation of motion for the nth particle in Fig. 1(b) as

$$m\ddot{u}_n = F_n,\ n = 1, 2, 3, \ldots, \tag{6}$$

where F_n is the total force exerted on this particle by the springs and dash pots that link it, respectively, with particle $(n-1)$ and particle $(n+1)$. To obtain the value of F_n, we use the following notation: F_{nr} and F_{nl} stand for the forces exerted on the nth particle from the right and the left sides, respectively; ε_{sr}, ε_{sl}, ε_{vr} and ε_{vl} designate the relative displacement of the closest right spring, left spring, right dash pot and left dash pot to the nth particle, respectively. In terms of particle displacements we have

$$\varepsilon_{sr} + \varepsilon_{vr} = u_{n+1} - u_n, \tag{7a}$$

$$\varepsilon_{sl} + \varepsilon_{vl} = u_n - u_{n-1}. \tag{7b}$$

With this notation, a free body diagram yields

$$F_n = F_{nr} - F_{nl}, \tag{8a}$$

with

$$F_{nr} = k\varepsilon_{sr} \equiv \gamma\dot{\varepsilon}_{vr}, \tag{8b}$$

$$F_{nl} = k\varepsilon_{sl} \equiv \gamma\dot{\varepsilon}_{vl}, \tag{8c}$$

where k designates the spring constant and γ is the dash pot's viscosity. Equations (7) and (8) may be combined to obtain

$$\frac{\dot{F}_n}{k} + \frac{F_n}{\gamma} = \dot{\varepsilon}_n, \tag{9}$$

where

$$\varepsilon_n = u_{n+1} - 2u_n + u_{n-1}. \tag{10}$$

Substituting for F_n from (6) into (9), using (10) and the fact that the chain is assumed to be initially at rest, yield the equation of motion

$$\ddot{u}_n + 2\alpha\dot{u}_n = \omega^2(u_{n+1} - 2u_n + u_{n-1}), \tag{11}$$

where

$$\alpha = k/2\gamma, \qquad \omega^2 = k/m. \tag{12}$$

Equation (11) together with the initial and boundary conditions (1b, c) completely describe the motion of the chain.

When α vanishes, equation (11) reduces to equation (1a). For nonvanishing α, however, the analysis used in [6] leads to

$$u_n(t) = \mathscr{L}^{-1}\left[\bar{\phi}(p)\left\{\frac{[(p+\alpha)^2 - \alpha^2]^{1/2}}{2\omega} - \left[\frac{(p+\alpha]^2 - \alpha^2}{4\omega^2} + 1\right]^{1/2}\right\}^{2n}\right]. \tag{13}$$

Notice that the expression (13), except for $\bar{\phi}(p)$, can be obtained from the corresponding expression for the nonviscous case (5a) if we replace p by $(p^2 + 2\alpha p)^{1/2}$.

We may invert (13) by recourse to the convolution theorem, the shifting rule, and published tables of Laplace transforms (see for example [12]†)

$$\mathscr{L}^{-1}\{\bar{g}[(p^2 - \beta^2)^{1/2}]\} = g(t) + \beta\int_0^t g(s)\,\frac{I_1[\beta(t^2 - s^2)^{1/2}]}{(t^2 - s^2)^{1/2}}\, s\, \mathrm{d}s, \tag{14}$$

where I_1 is the Bessel function of the second kind. For the present problem, $g(t)$ is the inverse transform of (13) when $\alpha \equiv 0$ and $\bar{\phi}(p) \equiv 1$; it is given in [6] as

$$g(t) = \frac{2n}{t}\, J_{2n}(2\omega t), \qquad n = 1, 2, 3, \ldots \tag{15}$$

Thus, (13–15) finally yield the general solution

$$u_n(t) = 2n\int_0^t \phi(t-\tau)e^{-\alpha\tau}\left[\frac{1}{\tau}J_{2n}(2\omega\tau) + \alpha\int_0^\tau J_{2n}(2\omega s)\,\frac{I_1[\alpha(\tau^2 - s^2)^{1/2}]}{(\tau^2 - s^2)^{1/2}}\,\mathrm{d}s\right]\mathrm{d}\tau. \tag{16}$$

For the special case of a constant boundary velocity V, $\phi(t) = Vt$, and (16) takes the form

$$\dot{u}_n(t) = 2nV\int_0^t e^{-\alpha\tau}\left[\frac{1}{\tau}J_{2n}(2\omega\tau) + \alpha\int_0^\tau J_{2n}(2\omega s)\,\frac{I_1[\alpha(\tau^2 - s^2)^{1/2}]}{(\tau^2 - s^2)^{1/2}}\,\mathrm{d}s\right]\mathrm{d}\tau. \tag{17}$$

† In [12] Formula (1.1.61) incorrectly omits β from the argument of the Bessel function. We also note a basic difference between the table's definition of the Laplace transform and that used here; in [12] and [13], Carson transform is used, i.e. in our notation $p\bar{g}(p) \to \bar{g}(p)$.

3.1 *The continuum limit*

A continuum description of the lattice may be obtained in the limit as the lattice spacing tends to zero. If we assume that u_{n+1} and u_{n-1} admit Taylor series expansions (assumed asymptotic) about $u_n \equiv u$ then, to the first order of approximation, equation (11) reduces to

$$\ddot{u} + 2\alpha\dot{u} - \omega^2 a^2 u'' = 0. \tag{18}$$

Here, "a" is the lattice spacing (assumed vanishingly small), and superposed primes denote differentiation with respect to the continuous x-coordinate; in equation (18), $x = na$. Referring to (14) and taking the Laplace transform of (18), it is easily found that the function $g(t)$ associated with (18) is now given by

$$g(t) = \delta(t - z), \tag{19}$$

where $z = x/c$, $c = \omega a$ is the phase velocity of the continuum, and $\delta(-)$ is the dirac delta function.

Replacing (15) by (19), the solution (16) now reduces to

$$u(x, t) = \phi(t - z)e^{-\alpha z} + \alpha z \int_z^t \phi(t - \tau)e^{-\alpha\tau} \frac{I_1[\alpha(\tau^2 - z^2)^{1/2}]}{(\tau^2 - z^2)^{1/2}} \, d\tau \quad \text{for} \quad t > z. \tag{20}$$

In particular, (17) reduces to

$$\frac{\dot{u}(x, t)}{V} = e^{-\alpha z} + \alpha z \int_z^t e^{-\alpha\tau} \frac{I_1[\alpha(\tau^2 - z^2)^{1/2}]}{(\tau^2 - z^2)^{1/2}} \, d\tau, \qquad t > z. \tag{21}$$

3.2 *Discussion and numerical results*

Inspection of the solution (21) shows that it consists of a wave propagating in the x-direction with velocity $c = \omega a$. For a fixed x, it also shows that the velocity is discontinuous at the wave front. The magnitude of this discontinuity can be obtained by setting $t = z$ in (21), resulting in

$$[\dot{u}(x, t)] = Ve^{-\alpha z}, \tag{22}$$

where the square brackets denote the jump discontinuity. For $t > z$, the velocity varies according to (21); its final value as $t \to \infty$ being the boundary value† $\dot{u}(0, t) = V$.

Using a Gaussian quadrature routine, the general integral in (17), together with the solution obtained for the continuous model (21), were evaluated as functions of time at the location of the 10th mass, for a varying number of α's. Results are depicted in Fig. 2. The parameter ω is chosen arbitrarily as 0·5. Examination of this figure reveals the interesting phenomena that the effect of discretization in the system is to introduce oscillations about the continuous system's behavior. The amplitudes of the oscillations damp out faster for larger α's. In general, as $t \to \infty$, the solutions of the continuous and the corresponding discrete systems asymptotically approach the limiting (as $t \to \infty$) value of the boundary input.

† This asymptotic conclusion can be shown by using the shifting formula, the tabulated inverse Laplace transform formula (see [14], p. 329, No. 93), and the limit as the transform parameter $p \to 0$.

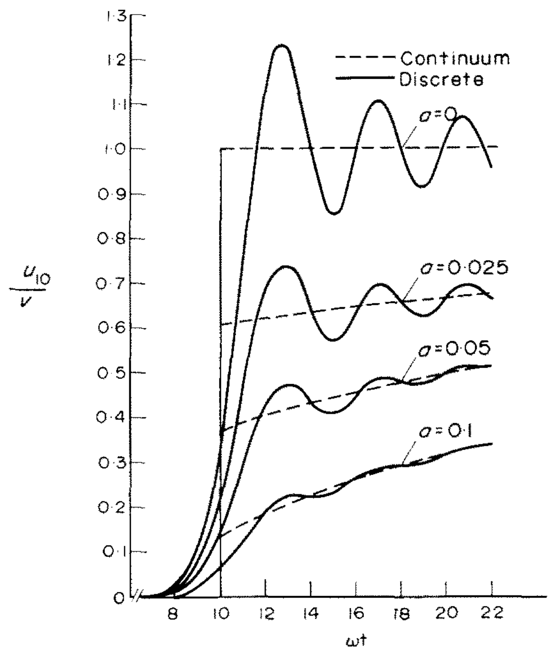

Fig. 2. Transient results for the constant boundary velocity.

4. DASH POTS IN PARALLEL

For this case we have

$$\varepsilon_{sr} = \varepsilon_{vr} \equiv u_{n+1} - u_n;\ \varepsilon_{sl} = \varepsilon_{vl} \equiv u_n - u_{n-1}, \tag{23}$$

and each spring and dash pot supports a different force. Considering again nearest-neighbor-interactions, the forces F_{nr} and F_{nl} (see Fig. 1c) are now given by

$$F_{nr} = k\varepsilon_{sr} + \gamma\dot{\varepsilon}_{vr} \equiv \left(k + \gamma\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{n+1} - u_n), \tag{24}$$

$$F_{nl} = k\varepsilon_{sl} + \gamma\dot{\varepsilon}_{vl} \equiv \left(k + \gamma\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_n - u_{n-1}). \tag{25}$$

Equations (8a) and (23–25) yield

$$F_n = \left(k + \gamma\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{n-1} - 2u_n + u_{n+1}), \tag{26}$$

where $\mathrm{d}/\mathrm{d}t$ stands for the time derivative.

Combining equations (26) and (6) results in

$$\ddot{u}_n = \omega^2\left(1 + \beta\frac{\mathrm{d}}{\mathrm{d}t}\right)(u_{n+1} - 2u_n + u_{n-1}), \tag{27}$$

where $\beta = \gamma/k$ and ω^2 is given in (12). Equation (27) and the initial and boundary conditions (1b, c) completely define the motion of the chain.

The method used in [6] now yields

$$\frac{\bar{u}_n(p)}{\bar{\phi}(p)} = \left[\frac{p}{2\omega(1+\beta p)^{1/2}} - \left(\frac{p^2}{4\omega^2(1+\beta p)} + 1\right)^{1/2}\right]^{2n}. \tag{28}$$

Again, the right hand side of (28) can be obtained from the corresponding expression for the problem solved in [6] if we replace p by $p/(1+\beta p)^{1/2}$. Comparatively speaking, the Laplace transform inversion process of the expression (28) is algebraically more involved and leads to solutions that are more complicated than the treatment of (13).

In order to obtain the inverse of (28), we use the powerful theorem (see [13]) which states: "If the transform of $g(t)$ is $\bar{g}(p)$ and that of $K(t, \eta)$ is $y(p)e^{-\eta v(p)}$, then the transform of

$$\int_0^\infty K(t, \eta) g(\eta)\,\mathrm{d}\eta \quad \text{is} \quad y(p)\bar{g}(v(p)), \tag{29}$$

where $y(p)$ and $v(p)$ are continuous functions of p, independent of η, $\mathrm{Re}[v(p)] \geq p_0 > 0$, η real and ≥ 0." For the present problem, $v(p) = p/(1+\beta p)^{1/2}$, $g(t)$ is the function given in (15), and $K(t, \eta)$ is to be determined.

For the special case $y(p) = 1/p$, $K(t, \eta)$ represents the solution for the propagation of a unit step displacement function that is applied to the free surface of a semi-infinite one-dimensional viscoelastic (of Voigt type) medium[15]; it is given by

$$K(t, \eta) = \int_\xi^\infty I_0[2(\xi|s-\xi|)^{1/2}]\left\{\frac{1}{\sqrt{\pi\tau}}\,\mathrm{Exp}\left(-\frac{s^2}{4\tau} - \tau\right) + \frac{1}{2}\left[e^{-s}\,\mathrm{erfc}\left(\frac{s}{2\sqrt{\tau}} - \sqrt{\tau}\right) - e^{s}\,\mathrm{erfc}\left(\frac{s}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right]\right\}\mathrm{d}s, \tag{30}$$

where

$$\xi = \eta/\beta, \qquad \tau = t/\beta,$$

and $\mathrm{erfc}(x)$ is the error function defined as

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,\mathrm{d}u. \tag{31}$$

The solution (30) yields the results: $K(t, 0) = 1$, $K(\infty, \eta) = 1$ and $K(0, \eta) = 0$. Its full spectrum is depicted in Fig. 3. Thus, for $\bar{\phi}(p) = 1/p$, which corresponds to the unit step boundary condition $u(t) = H(t)$, the response of the lattice is given by

$$u_n(t) = 2n\int_0^\infty K(t, \eta) J_{2n}(2\omega\eta)\,\frac{\mathrm{d}\eta}{\eta}. \tag{32}$$

The response of the lattice to the general boundary condition (1b) can be obtained by first differentiating (32) with respect to time then convoluting the result with $\phi(t)$. In this manner, we get

$$u_n(t) = 2n\int_0^t \phi(t-q)\left\{\int_0^\infty K^*(q, \eta) J_{2n}(2\omega\eta)\,\frac{\mathrm{d}\eta}{\eta}\right\}\mathrm{d}q, \tag{33}$$

where

$$K^*(t, \eta) = \frac{\partial}{\partial t} K(t, \eta).$$

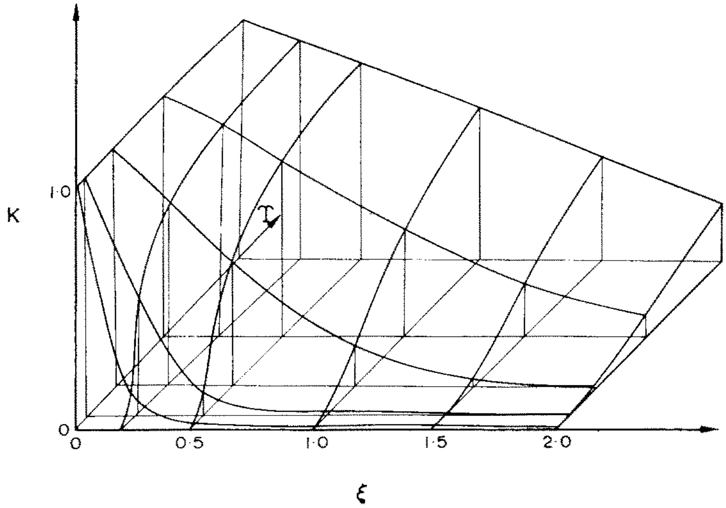

Fig. 3. A spectrum of the solution (30).

For the corresponding continuum model, i.e. when the lattice spacing tends to zero, equation (19) again replaces (15) and thus reduces (32) to (30) with z replacing η.

4.1 *Asymptotic results and applications to composites*

In the previous section, exact solutions for equation (27) were obtained in terms of the infinite integrals (32) and (33). However, for their possible applications as mathematical models of the more complicated systems such as periodic elastic and viscoelastic composites (whose exact analytical solutions cannot generally be obtained), it is convenient to seek asymptotic solutions of equation (27). Simple asymptotic results of (27) can be obtained by seeking solutions valid at long times, for example, times that are of the same order as the arrival time of the disturbance at a fixed station far from the boundary mass.

To this end, we rewrite equation (28) as

$$\bar{u}_n(p) = \bar{\phi}(p)\exp[-2n\sinh^{-1}(p^*/2\omega)] \tag{34}$$

where $p^* = p/(1+\beta p)^{1/2}$. Now, for small values of p, with np/ω not necessarily small, we may expand $\sinh^{-1}(p^*/2\omega)$ in terms of integral powers of p, and to $0(p^3)$ obtain the equivalent expression

$$\bar{u}_n(p) = \bar{\phi}(p)\exp\left[\frac{-np/\omega}{1+\tau_1 p+\tau_2 p^2}+\cdots\right], \tag{35}$$

where

$$\tau_1 = \beta/2 \qquad \text{and} \qquad \tau_2 = 1/24\omega^2 - \beta^2/8. \tag{36}$$

A similar result to (35) can also be obtained by utilizing an equivalent alternative asymptotic scheme in which imposed signal wave lengths are assumed large compared with the lattice spacing. In fact, if we define the quantities

$$x = na, \qquad c = \omega a, \qquad x = \xi l$$
$$t = l\tau/c, \qquad \varepsilon = a/l$$

where "a" is the lattice spacing, c is the phase velocity when $\varepsilon = 0$, l is a nondimensionalizing macrodimension which, for example, may represent the wave length of an imposed signal, and ε is a nondimensional quantity, then for small values of ε, a Taylor series expansion reduces (27) to the global partial differential equation

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \xi^2} - \beta\omega\varepsilon \frac{\partial^3 u}{\partial \xi^2 \partial \tau} - \frac{\varepsilon^2}{12}\frac{\partial^4 u}{\partial \xi^4} + 0(\varepsilon^3) = 0 \tag{37}$$

which obviously is valid far away from the boundary $\xi = 0$. This equation has the same form as the linear "viscous" dispersion model proposed by Bade *et al.* for composites [16]. In the purely elastic case, β vanishes and (37) takes the form of the so-called "global" partial differential equations derived in [7, 8, 17] to describe the response of a variety of periodic elastic composites. These global equations are found to differ only in the material properties and geometric arrangement dependence of their coefficients.

While the solution of (37) is important in demonstrating the qualitative response of composites, the exact response of an actual composite cannot be obtained unless the corresponding coefficients of this differential equation are derived. This is important since for some periodic composites (see [7]) it was demonstrated that not only the values of the coefficients in (37) change but the sign of the highest order derivative also changes. This resulted in a somewhat different character of this composite's response. However, this response can also be deduced from the solution of (37) when the sign of its highest order derivative is changed.

Taking Laplace transform of (37) with respect to τ, followed by assuming exponential solutions of the form $\bar{u}_n(s) = A(s)e^{-\alpha\xi}$, then expanding for small values of εs, finally yield

$$\bar{u}_n(s) = \bar{\phi}(s\omega\varepsilon)\exp\left[\frac{-s\xi}{1 + \tau_1\omega\varepsilon s + \tau_2\omega^2\varepsilon^2 s^2}\right] \tag{38}$$

which is (35) in dimensional form.

In order to find the inverse of (35), we utilize the tabulated formula (see [12], formula (2–4.64, 65), (1–1.38), and see also footnote following equation (17)).

$$\mathscr{L}^{-1}\left[\exp\left[-\frac{1}{a(p+b)}\right] - 1\right] = -\frac{e^{-bt}}{\sqrt{at}} J_1(2\sqrt{t/a}), \tag{39}$$

$$\mathscr{L}^{-1}\left[\frac{1}{p}\bar{f}\left(p + \frac{c}{p}\right)\right] = \int_0^t J_0[2\sqrt{eu(t-u)}]f(u)\,\mathrm{d}u. \tag{40}$$

Using (39) and (40) consecutively, and after some algebraic manipulation, the inverse of (35) can finally be obtained for the special choice $\bar{\phi}(p) = 1/p$ as

$$u_n(t) = 1 - \int_0^{2\sqrt{nt/\omega\tau_2}} \exp(-\tau_1\omega u^2/4n) J_1(u)\, J_0\left[\frac{u}{2n}\{\omega(4nt - \omega\tau_2 u^2\}^{1/2}\right]\mathrm{d}u. \tag{41}$$

If the sign of the last term in (37) changes (corresponding to the sign change of τ_2 in (35)) then (41) still holds with I_0 replacing J_0.

4.2 *Discussion and numerical results*

Compared with the response of a homogeneous elastic medium to a unit step function, the integral term in the solution (41) represents the compound influence of the discrete and viscous properties of the present lattice. The parameter τ_1 accounts for viscosity, while, as can be seen from (36), the parameter τ_2 is associated with the influence of both the discretization and viscosity. The vanishing of τ_1 and τ_2 (also equivalent to $p \to 0$) reduces (41) to

$$u_n(t) = 1 - \int_0^\infty J_1(s)J_0[s(\omega t/n)^{1/2}]\,\mathrm{d}s \tag{42}$$

which is precisely the unit step function (see [18], p. 667, No. 3)

$$u_n(t) = H\left(t - \frac{n}{\omega}\right), \tag{43}$$

a result that may also be easily inferred from (35).

If τ_1 vanishes and $\tau_2 > 0$, the spatial periodicity produces superposed oscillations about the unit step boundary input. Up to the present order of approximation, these solutions are equivalent to the solutions in terms of trigonometric functions and Airy integral (Head of the Pulse) obtained in [17, 8] and [1], respectively. If τ_1 vanishes and τ_2 is preceded by a negative sign, again (41) holds with $\tau_1 = 0$ and I_0 replacing J_0. For this case, the present solution is equivalent to the Head of the Pulse approximation obtained by Sve[9] and Nayfeh and Gurtman[7] for the quasi-shear pulse motion in a laminated composite. The pulse now is preceded by oscillations and then a smooth rise to the steady value.

For the general case of nonvanishing τ_1 and τ_2, the viscosity plays two roles; (i) it damps the amplitudes of the oscillations, and (ii) increases their frequency due to the fact that it tends to decrease the value of τ_2. This latter role of viscosity is quite important because it effectively counters the effect of discretization. In fact, if one suppresses the exponential term, the resulting solution in (41) can be brought arbitrarily close to the unit step function merely by increasing the "viscosity" which, in turn, decreases the value of τ_2.

Using a combination of the Romberg and the Curtis–Clenshaw quadrature routines, the exact solution (32) was integrated numerically for $n = 10$, $\omega = 0{\cdot}5$ and for the two values of the viscosity parameter, $\beta = 0$ and $\beta = 0{\cdot}2$. Results for the unit displacement boundary input vs time are shown in Fig. 4 as solid lines. For the sake of comparison, the corresponding

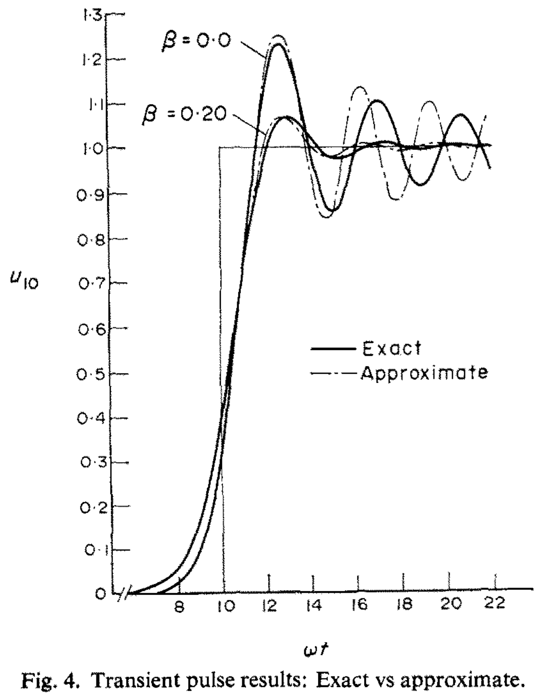

Fig. 4. Transient pulse results: Exact vs approximate.

approximate solution (42) is also shown on the same figure in broken lines. The comparison shows excellent correlation of the exact and approximate results, particularly near the wave front. This behavior is also typical of the existing correlation between exact (ray-theory numerical calculations) and approximate solutions obtained for disturbance propagation normal to the layers of a laminated composite by Hegemier and Nayfeh[17]. Finally, results from (42) showing the relative influence of β are shown in Fig. 5.

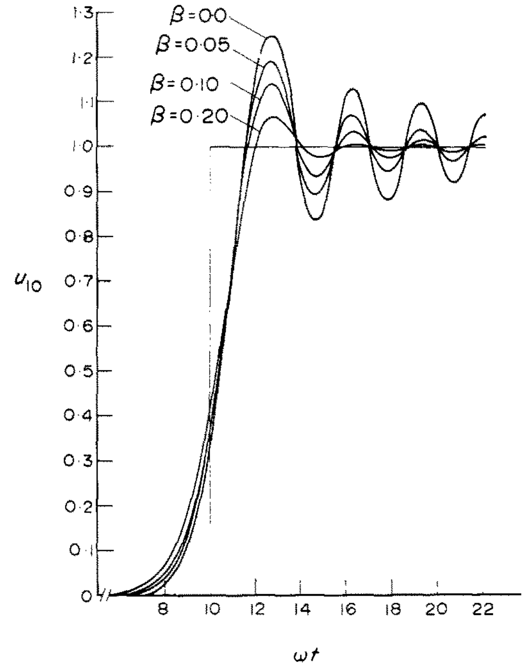

Fig. 5. Spectrum of viscosity influence on transient pulses.

Acknowledgements—The author thanks Mr. E. W. Sims and Mr. A. J. Good of Systems, Science and Software for their help in some of the numerical calculations.

REFERENCES

1. J. C. Peck and G. A. Gurtman, Dispersive pulse propagation parallel to the interfaces of a laminated composite, *J. appl. Mech.* **35**, 479 (1969).
2. D. S. Drumheller and H. J. Sutherland, A lattice model for composite materials, *J. appl. Mech.* **40**, 149 (1973).
3. L. Brillouin, *Wave Propagation in Periodic Structures*. Dover (1953).
4. R. Manvi, G. E. Duvall and S. C. Lowell, Finite amplitude longitudinal waves in lattices, *Int. J. Mech. Sci.* **11**, 1 (1969).
5. R. Weinstock, Propagation of a longitudinal disturbance on a one-dimensional lattice, *Am. J. Phys.* **38**, 1289 (1970).
6. Adnan H. Nayfeh, and M. H. Rice, On the propagation of disturbances in a semi-infinite one-dimensional lattice, *Am. J. Phys.* **40**, 469 (1972).
7. Adnan H. Nayfeh and G. A. Gurtman, A continuum approach to the propagation of shear waves in laminated wave guides, to appear in *J. appl. Mech.* (1974). ASME Paper No. 73—APMW—12.
8. G. A. Hegemier, G. A. Gurtman and Adnan H. Nayfeh, A continuum mixture theory of wave propagation in laminated and fiber reinforced composites, *Int. J. Solids Struct.* **9**, 395 (1973).

9. C. Sve, Propagation of a shear pulse parallel to the interfaces of a periodically laminated medium, *J. appl. Mech.* **37**, 203 (1970).
10. T. E. Michaels, *et al.*, Experimental Wave Propagation Studies in 1-D FRC Materials, Interim Report. General Motors Corporation, Manufacturing Development, Materials and Structures Laboratory, MSL-71-16. May (1971).
11. C. Sve, Stress-wave attenuation in composite materials, *J appl. Mech.* **39**, 1151 (1972).
12. V. A. Ditkin, and A. P. Prudnikov, *Formulaire Pour Le Calcul Operationnel*, edited by Musson and Cie, Paris (1967).
13. N. W. McLachlan, *Laplace Transforms and Their Applications to Differential Equations*. Dover (1962)
14. R. Churchill, *Operational Mathematics*. McGraw-Hill (1958).
15. I. N. Zvyeryev, Propagation of disturbances in a visco-elastic and viscoplastic Rod, *Prikl. Mat. Mekha.* **14**, 1950, (in Russian).
16. W. L. Bade, A. T. Kramer, R. R. McMath and J. M. Yos, *Theoretical and Experimental Study of Advanced Materials* (TESAM)—I: *Shock Wave Dispersion in 3-D Reinforced Heat Shield Materials*, Air Force Weapons Laboratory, Kirtland AFB, New Mexico, AFWL-TR-70-91, (Vol. I). March (1971).
17. G. A. Hegemier and Adnan H. Nayfeh, A continuum theory for wave propagation in laminated composites. Case 1: propagation normal to laminates, *J. appl. Mech.* **40**, 503 (1973).
18. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*. p. 667. Academic Press (1965).

Абстракт — Исследуются дискретные вязкие решетки, с целью математического моделирования распространения общих возмущений в полу-бесконечных упругих и вязко-упругих периодических составных материалах. Применяется метод интегрального преобразования, для определения строгого и приближенного решений для этих случаев, в которых вязкость введена к упругой решетке путем добавления амортизаторов в оба ряды и параллельных к пружинам, соединяющих смежные частицы. Получаются, также, решения для сплошных моделей (для изчезающих параметров решетки), как непосредственно или в качестве специальных случаев. Указывается, что эффект периодической диспретизации в этих системах вызывает колебания около решений для сплошной среды. Сравнение строгого и приближенного решений проявляет отличную корреляцию, особенно вблизи фронта волны.